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A novel generalization of Bézier curve and surface

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Abstract

A new formulation for the representation and designing of curves and surfaces is presented. It is a novel generalization of Bézier curves and surfaces. Firstly, a class of polynomial basis functions with n adjustable shape parameters is present. It is a natural extension to classical Bernstein basis functions. The corresponding Bézier curves and surfaces, the so-called Quasi-Bézier (i.e., Q-Bézier, for short) curves and surfaces, are also constructed and their properties studied. It has been shown that the main advantage compared to the ordinary Bézier curves and surfaces is that after inputting a set of control points and values of newly introduced n shape parameters, the desired curve or surface can be flexibly chosen from a set of curves or surfaces which differ either locally or globally by suitably modifying the values of the shape parameters, when the control polygon is maintained. The Q-Bézier curve and surface inherit the most properties of Bézier curve and surface and can be more approximated to the control polygon. It is visible that the properties of end-points on Q-Bézier curve and surface can be locally controlled by these shape parameters. Some examples are given by figures.

MSC: primary 65D17;65D10; secondary 42A10

Keywords: Computer aided geometric design; Generalized Bernstein basis functions; Bézier curve/surface patches; Q-Bézier curve/surface patches; Shape parameters

1. Introduction

Parametric representation of curves and surfaces is widely used in Computer Aided Geometric Design (CAGD) and Computer Graphics (CG) to model interactively various surfaces by means of surface patches. When representing a parametric curve or surface, it is important which basis functions are used if we wish to preserve the shape of the curve or surface. For these reasons the Bernstein–Bézier curve and surface representation plays a significant role in CAGD&CG [5,14]. Many authors have studied several kinds of spline for curve and surface design and control [2,3,8–10,15,16,18,24]. In general, the common spline interpolation is the fixed interpolation which means that the shape of the interpolating curve or surface is fixed for the given interpolating data and control polygon, since the interpolating function is unique for the given control points. If one wishes to modify the shape of the interpolating curve, the interpolating data need to be changed. An important question is how can the shape of the curve be modified

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under the condition that the given data and control polygon are not changed? That is a important problem in CAGD&CG. Theoretically speaking, it is contradictory to the uniqueness of the interpolating function for the given interpolating data and control polygon.

The rational Bézier model is a powerful tool for constructing free-form curves and surfaces. In recent years, the rational spline with parameters has received attention in the literature [4,6,11]. For the given interpolating data, the change of the parameters causes the change of the interpolating curve, so that the interpolating curve may be modified to be the needed shape if suitable parameters exist. That is, the uniqueness of the interpolating function for the given data is replaced by the uniqueness of the interpolating curve for the given data and the selected parameters. However, it has many shortcomings due to its rational form and nonpolynomial form. For instance, repeated differentiation produces curves of very high degree.

Bernstein polynomials and Bézier curves are of fundamental importance for CAGD &CG. They are used for the design of curves and they are the starting point for several generalizations [1,6,7,12,13,17,19–23,25,26]: in particular to higher dimensions and to B-splines. Powerful algorithms are available for both their algebraic construction and their visualization, and their basic theory (explained beautifully in Farin's book [5]) has been examined repeatedly from different new angles: see, e.g., for the "basis function" point of view, many bases are presented in new spaces other than the polynomial space [12,13,19,20,22,25,26], [1] for the "natural generalization of Bézier curves", [7] for "barycentric" and [21] for "blossoming". Although these algorithms are very effective and widely used in practice, they have a drawback: they are not able to control the shape of the corresponding curves keeping the control polygon unchanged.

In the present paper, we introduce an approach to the generalization of Bernstein polynomials and Bézier curves and surfaces which seems to be entirely new. It is based on the novel ideas that the shape of the curves and surfaces is controlled by the control edges of the control polygon, not by the control points. These curves and surfaces have more simplify form and clear geometric meaning than rational Bézier curves and surfaces.

The present paper is organized as follows. In Section 2, a class of polynomial basis functions with adjustable shape parameters is constructed and the properties of the basis functions are shown. The corresponding Q-Bézire curves and surfaces are presented and their properties studied in Sections 3 and 6, respectively. In Section 4 the effect of the shape of the curve by the shape parameters is given. The composite Q-bézier curves are discussed in Section 5 as well.

2. Generalized Bernstein basis function

Firstly, the definition of generalized Bernstein basis functions is constructed as follows.

Definition 2.1. For every nature number $n (n \geqslant 2)$ and n arbitrarily selected real values of $\lambda_i, i = 1, 2, \ldots, n$, the following polynomial functions

$$\begin{cases} b_0^n(t) = (1-t)^n(1-\lambda_1 t), \\ b_i^n(t) = t^i(1-t)^{n-i}\left(\binom{n}{i} + \lambda_i - \lambda_i t - \lambda_{i+1} t\right), \quad i = 1, \ldots, \left[\frac{n}{2}\right] - 1, \\ b_{[\frac{n}{2}]}^n(t) = t^{[\frac{n}{2}]}(1-t)^{n-[\frac{n}{2}]}\left(\binom{n}{[\frac{n}{2}]} + \lambda_{[\frac{n}{2}]} - \lambda_{[\frac{n}{2}]} t + \lambda_{[\frac{n}{2}]+1} t\right), \\ b_i^n(t) = t^i(1-t)^{n-i}\left(\binom{n}{i} - \lambda_i + \lambda_i t + \lambda_{i+1} t\right), \quad i = \left[\frac{n}{2}\right] + 1, \ldots, n-1, \\ b_n^n(t) = t^n(1 - \lambda_n + \lambda_n t) \end{cases} \tag{2.1}$$

are called the generalized Bernstein polynomials of degree n associated with the shape parameters $\{\lambda_i\}_{i=1}^n$, where $\lambda_i \in \left[-\binom{n}{i}, \binom{n}{i-1}\right], i = 1, 2, \ldots, [\frac{n}{2}]$, $\lambda_i \in \left[-\binom{n}{i-1}, \binom{n}{i}\right], i = [\frac{n}{2}] + 1, \ldots, n$ and $[\frac{n}{2}] = \begin{cases} \frac{n}{2}, & \textit{if } n \textit{ is even}, \\ \frac{n+1}{2}, & \textit{if } n \textit{ is odd}. \end{cases}$

It is obvious, for $\lambda_i = 0, i = 1, 2, \ldots, n$, the basis functions are classical Bernstein polynomials of degree n.

Theorem 2.1. *The generalized Bernstein basis functions of degree n associated with the shape parameters* $\{\lambda_i\}_{i=1}^n$ (2.1) *have the following properties*:

(a) *nonnegativity*: $b_i^n(t) \geqslant 0$, $i = 0, 1, \ldots, n$;
(b) *partition of unity*: $\sum_{i=0}^n b_i^n(t) \equiv 1$;
(c) *symmetry*: *for* $i = 0, 1, \ldots, n$;

$$b_i^n\left(t; \lambda_1, \ldots, \lambda_{[\frac{n}{2}]-1}, \lambda_{[\frac{n}{2}]}, \lambda_{[\frac{n}{2}]+1}, \ldots, \lambda_n\right) = b_{n-i}^n\left(1-t; \lambda_n, \ldots, \lambda_{[\frac{n}{2}]+1}, (-1)^n \lambda_{[\frac{n}{2}]}, \lambda_{[\frac{n}{2}]-1}, \ldots, \lambda_1\right).$$

Proof. (a) For $t \in [0, 1]$ and the range of λ_i, $i = 1, \ldots, n$ in Definition 2.1, then $1 - \lambda_1 t \geqslant 0$, $\binom{n}{i} + \lambda_i - \lambda_i t - \lambda_{i+1} t \geqslant 0$, $\binom{n}{[\frac{n}{2}]} + \lambda_i - \lambda_i t + \lambda_{i+1} t \geqslant 0$, $\binom{n}{i} - \lambda_i + \lambda_i t + \lambda_{i+1} t \geqslant 0$ and $1 - \lambda_n t \geqslant 0$. It is obvious that $b_i^n(t) \geqslant 0$, $i = 0, 1, \ldots, n$.

(b) $\sum_{i=0}^n b_i^n(t) = \sum_{i=0}^n \binom{n}{i} t^i (1-t)^{n-i} \equiv 1$.

(c) If n is even, the symmetry of $\{b_i^n(t)\}_{i=0}^n$ holds obviously.

For n is odd,

$$-\binom{n}{[\frac{n}{2}]-1} \leqslant -\lambda_{[\frac{n}{2}]} \leqslant \binom{n}{[\frac{n}{2}]},$$

because $n - [\frac{n}{2}] = [\frac{n}{2}] - 1$, hence,

$$-\binom{n}{[\frac{n}{2}]} \leqslant -\lambda_{[\frac{n}{2}]} \leqslant \binom{n}{[\frac{n}{2}]-1}$$

and for $i = 0, 1, \ldots, n$,

$$b_i^n\left(t; \lambda_1, \ldots, \lambda_{[\frac{n}{2}]-1}, \lambda_{[\frac{n}{2}]}, \lambda_{[\frac{n}{2}]+1}, \ldots, \lambda_n\right) = b_{n-i}^n\left(1-t; \lambda_n, \ldots, \lambda_{[\frac{n}{2}]+1}, (-1)^n \lambda_{[\frac{n}{2}]}, \lambda_{[\frac{n}{2}]-1}, \ldots, \lambda_1\right).$$

Obviously when n is odd, the symmetry of $\{b_i^n(t)\}_{i=0}^n$ holds as well. $\square$

Fig. 1 shows the curves of the generalized polynomial basis functions for $n = 2$ and $\lambda_1 = -2$, $\lambda_2 = 1$ (dashed lines), $\lambda_1 = -1$, $\lambda_2 = -1$ (solid lines) and $\lambda_1 = 1$, $\lambda_2 = -2$ (dotted lines), respectively.

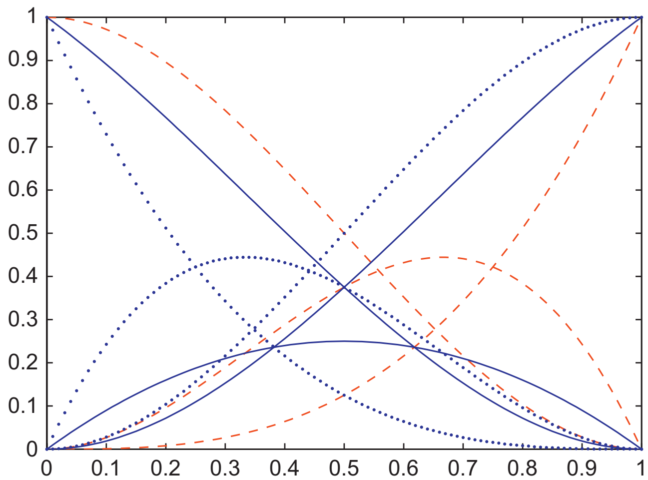

Fig. 1. The quadratic generalized Bernstein basis functions.

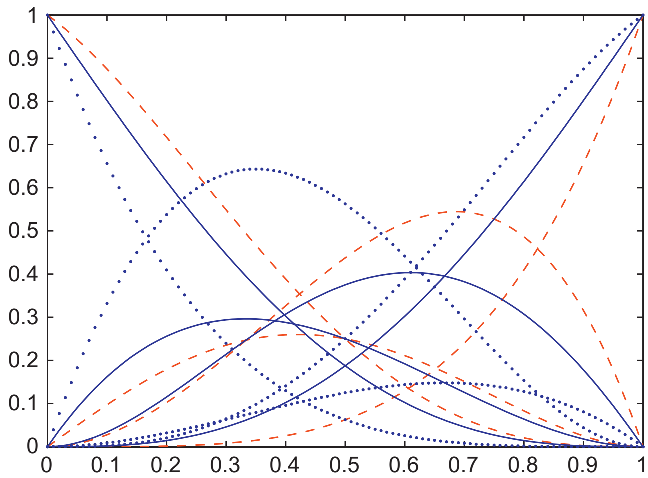

Fig. 2. The cubic generalized Bernstein basis functions.

Fig. 2 shows the curves of the generalized polynomial basis functions for $n = 3$ and $(\lambda_1, \lambda_2, \lambda_3) = (-2, 0, 1)$ (dashed lines), for $(\lambda_1, \lambda_2, \lambda_3) = (-1, 1, -1)$ (solid lines) and for $(\lambda_1, \lambda_2, \lambda_3) = (1, -2, -2)$ (dotted lines), respectively.

3. Construction of the Q-Bézier curve

Definition 3.1. Given points $\boldsymbol{P}_i\,(i = 0, 1, \cdots, n) \in \mathbb{R}^2$ or $\mathbb{R}^3$, then

$$\boldsymbol{r}(t) = \sum_{i=0}^{n} \boldsymbol{P}_i b_i^n(t), \qquad t \in [0, 1] \tag{3.1}$$

is called Q-Bézier curve with shape parameters, where $\lambda_i \in \left[-\binom{n}{i}, \binom{n}{i-1}\right]$, $i = 1, 2, \ldots, [\frac{n}{2}]$, $\lambda_i \in \left[-\binom{n}{i-1}, \binom{n}{i}\right]$, $i = [\frac{n}{2}] + 1, \ldots, n$.

From the definition of the basis functions (2.1), some properties of Q-Bézier curve can be obtained as follows:

Theorem 3.1. *The Q-Bézier curves* (3.1) *have the following properties*:

(a) *Terminal properties*:

$$\boldsymbol{r}(0) = \boldsymbol{P}_0, \quad \boldsymbol{r}(1) = \boldsymbol{P}_n. \tag{3.2}$$

Let $\boldsymbol{a}_i = \boldsymbol{P}_i - \boldsymbol{P}_{i-1}$, $i = 1, 2, \ldots, n$, *then*

$$\boldsymbol{r}'(0) = (n + \lambda_1)\boldsymbol{a}_1, \quad \boldsymbol{r}'(1) = (n + \lambda_n)\boldsymbol{a}_n, \tag{3.3}$$

$$\boldsymbol{r}''(0) = n(n-1)(\boldsymbol{a}_2 - \boldsymbol{a}_1) - 2n\lambda_1\boldsymbol{a}_1 + 2\lambda_2\boldsymbol{a}_2, \tag{3.4}$$

$$\boldsymbol{r}''(1) = n(n-1)(\boldsymbol{a}_n - \boldsymbol{a}_{n-1}) + 2n\lambda_n\boldsymbol{a}_n + 2\lambda_{n-1}\boldsymbol{a}_{n-1}. \tag{3.5}$$

(b) *Symmetry*: *Both* $\boldsymbol{P}_0, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$ *and* $\boldsymbol{P}_n, \boldsymbol{P}_{n-1}, \ldots, \boldsymbol{P}_0$ *define the same Q-Bézier curve in a different parametrization, i.e.*,

$$\boldsymbol{r}(t; \{\lambda_j\}_{j=1}^n; \{\boldsymbol{P}_i\}_{i=0}^n) = \boldsymbol{r}\left(1 - t; \lambda_n, \ldots, \lambda_{[\frac{n}{2}]+1}, (-1)^n\lambda_{[\frac{n}{2}]}, \lambda_{[\frac{n}{2}]-1}, \ldots, \lambda_1; \{\boldsymbol{P}_{n-i}\}_{i=0}^n\right). \tag{3.6}$$

(c) *Geometric invariance*: *The shape of a Q-Bézier curve is independent of the choice of coordinates*, *i.e.*, *Eq.* (3.1) *satisfies the following two equations*:

$$\begin{aligned}&\boldsymbol{r}(t;\{\lambda_j\}_{j=1}^n;\boldsymbol{P}_0+\boldsymbol{q},\boldsymbol{P}_1+\boldsymbol{q},\ldots,\boldsymbol{P}_n+\boldsymbol{q})=\boldsymbol{r}(t;\{\lambda_j\}_{j=1}^n;\boldsymbol{P}_0,\boldsymbol{P}_1,\ldots,\boldsymbol{P}_n)+\boldsymbol{q},\\&\boldsymbol{r}(t;\{\lambda_j\}_{j=1}^n;\boldsymbol{P}_0*\boldsymbol{T},\boldsymbol{P}_1*\boldsymbol{T},\cdots,\boldsymbol{P}_n*\boldsymbol{T})=\boldsymbol{r}(t;\{\lambda_j\}_{j=1}^n;\boldsymbol{P}_0,\boldsymbol{P}_1,\cdots,\boldsymbol{P}_n)*\boldsymbol{T},\end{aligned}\tag{3.7}$$

where $\boldsymbol{q}$ *is an arbitrary vector in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$, *and* $\boldsymbol{T}$ *is an arbitrary* $d\times d$ *matrix*, $d=2$ *or* 3.

(d) *Convex hull property*: *The entire Q-Bézier curve segment must lie inside its control polygon spanned by* $\boldsymbol{P}_0,\boldsymbol{P}_1,\ldots,\boldsymbol{P}_n$.

4. Shape control of the Q-Bézier curve

For $t\in(0,1)$, we rewrite (3.1) as follows:

$$\boldsymbol{r}(t)=\sum_{i=0}^{n}\boldsymbol{P}_iB_i^n(t)+\sum_{i=1}^{[\frac{n}{2}]}\lambda_i(1-t)^{n+1-i}t^i\boldsymbol{a}_i-\sum_{i=[\frac{n}{2}]+1}^{n}\lambda_i(1-t)^{n+1-i}t^i\boldsymbol{a}_i,\tag{4.1}$$

where $B_i^n(t)=\binom{n}{i}(1-t)^{n-i}t^i$, $i=0,1,\ldots,n$ is classical Bernstein polynomials of degree n.

Obviously, shape parameters $\lambda_i(i=1,2,\ldots,n)$ only affect curve on the corresponding control edges $\boldsymbol{a}_i(i=1,2,\ldots,n)$, i.e., the shape of the curve can be modified by the control edges of control polygon with altering the values of the corresponding shape parameters. In fact, from (4.1), we can also predict the following behaviors of the curves.

The shape parameters λ_i, $i=1,2,\ldots,n$ serve to local control in the curves: as $\lambda_i(i=1,2,\ldots,[\frac{n}{2}])$ increases, the curve moves in the direction of $\boldsymbol{a}_i(i=1,2,\ldots,[\frac{n}{2}])$, as $\lambda_i(i=1,2,\ldots,[\frac{n}{2}])$ decreases, the curve moves in the opposite direction of $\boldsymbol{a}_i(i=1,2,\ldots,[\frac{n}{2}])$. The same effects on the edge $\boldsymbol{a}_i(i=[\frac{n}{2}]+1,\ldots,n)$ played by the corresponding shape parameters $\lambda_i(i=[\frac{n}{2}]+1,\ldots,n)$.

The effect of the shape parameters of the Q-Bézier curves (3.1) is clear. Figs. 3–10 show the effects on shape of the curve with altering the values of $\lambda_i(i=1,2,\ldots,n)$ for $n=3$ and $n=4$, respectively.

In the cubic case (n=3), Figs. 3–6 are for the demonstration of shape under local controlled. The points $\boldsymbol{r}(\frac{1}{3})$, $\boldsymbol{r}(\frac{1}{2})$, $\boldsymbol{r}(\frac{2}{3})$ on the curve (3.1) are marked, respectively.

In Fig. 3, the curves are generated by setting $\lambda_2=\lambda_3=0$ and changing λ_1 to $\lambda_1=0$ (solid lines) and $\lambda_1=-1$ (dashed lines) and $\lambda_1=-2$ (dotted lines) and $\lambda_1=1$ (dashdotted lines), respectively.

In Fig. 4, the curves are generated by setting $\lambda_1=\lambda_3=0$ and changing λ_2 to $\lambda_2=0$ (solid lines) and $\lambda_2=-1$ (dashed lines) and $\lambda_2=-2$ (dotted lines) and $\lambda_2=1$ (dashdotted lines) , respectively.

In Fig. 5, the curves are generated by setting $\lambda_1=\lambda_2=0$ and changing λ_3 to $\lambda_3=0$ (solid lines) and $\lambda_3=-1$ (dashed lines) and $\lambda_3=-2$ (dotted lines) and $\lambda_3=1$ (dashdotted lines), respectively.

In Fig. 6, the curves are generated by changing $(\lambda_1,\lambda_2,\lambda_3)$ to $(-3,-2,-1)$ (solid lines) and $(-2,-3,-2)$ (dashed lines) and $(-1,-1,-1)$ (dotted lines) and $(1,1,0)$ (dashdotted lines), respectively.

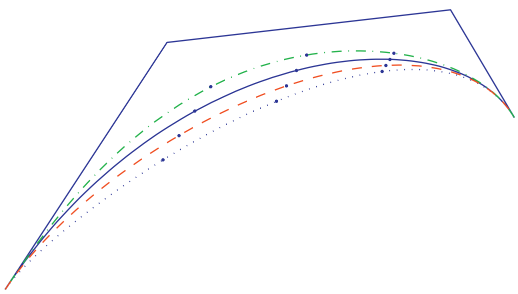

Fig. 3. The effect of the shape of cubic Q-Bézier curves by λ_1.

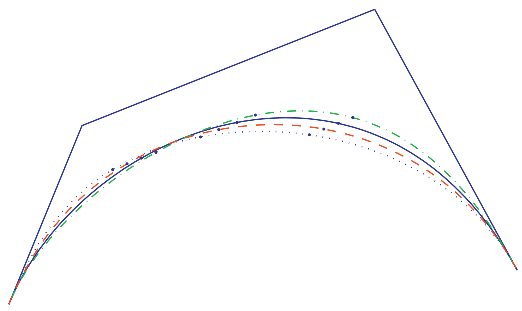

Fig. 4. The effect of the shape of cubic Q-Bézier curves by λ_2.

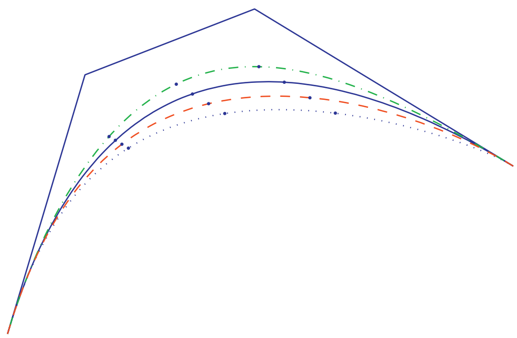

Fig. 5. The effect of the shape of cubic Q-Bézier curves by λ_3.

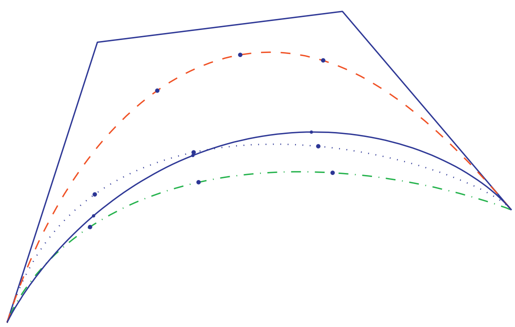

Fig. 6. The effect of the shape of cubic Q-Bézier curves by λ_1, λ_2, λ_3.

In the quartic case ($n = 4$), Figs. 7–10 are for the demonstration of shape under local controlled. The points $\boldsymbol{r}(\frac{1}{4}), \boldsymbol{r}(\frac{1}{2}), \boldsymbol{r}(\frac{3}{4})$ on the curve (3.1) are marked, respectively.

In Fig. 7, the curves are generated by setting $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and changing λ_1 to $\lambda_1 = 0$ (solid lines) and $\lambda_1 = -4$ (dotted lines) and $\lambda_1 = -2$ (dashdotted lines) and $\lambda_1 = 1$ (dashed lines), respectively.

In Fig. 8, the curves are generated by setting $\lambda_1 = \lambda_3 = \lambda_4 = 0.5$ and changing λ_2 to $\lambda_2 = 0$ (solid lines) and $\lambda_2 = -4$ (dotted lines) and $\lambda_1 = -2$ (dashdotted lines) and $\lambda_2 = 1$ (dashed lines), respectively.

In Fig. 9, the curves are generated by setting $\lambda_1 = \lambda_2 = \lambda_4 = 1$ and changing λ_3 to $\lambda_3 = 0$ (solid lines) and $\lambda_3 = -4$ (dotted lines) and $\lambda_3 = -2$ (dashdotted lines) and $\lambda_3 = 1$ (dashed lines), respectively.

In Fig. 10, the curves are generated by setting $\lambda_1 = -2$, $\lambda_2 = -3$, $\lambda_3 = -2$ and changing λ_4 to $\lambda_4 = 0$ (solid lines) and $\lambda_4 = -4$ (dotted lines) and $\lambda_4 = -2$ (dashdotted lines) and $\lambda_4 = 1$ (dashed lines), respectively.

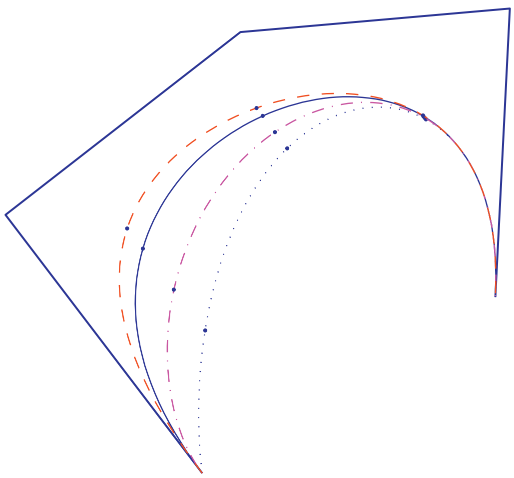

Fig. 7. The effect of the shape of quartic Q-Bézier curves by λ_1.

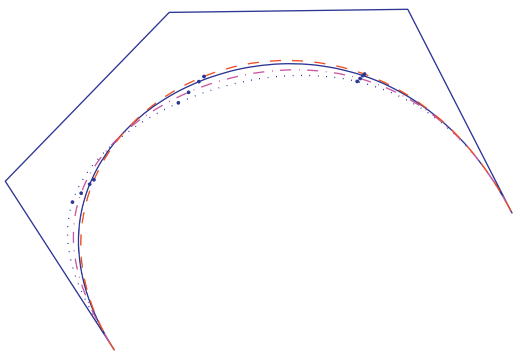

Fig. 8. The effect of the shape of quartic Q-Bézier curves by λ_2.

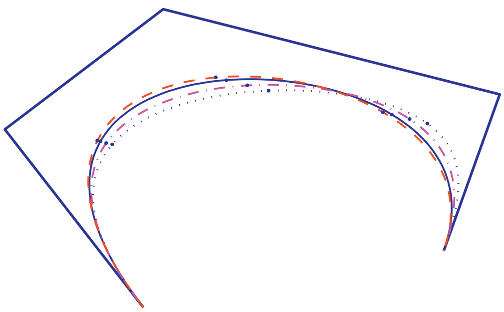

Fig. 9. The effect of the shape of quartic Q-Bézier curves by λ_3.

In order to construct closed Q-Bézier curve, we can set $\boldsymbol{P}_n = \boldsymbol{P}_0$. For $n = 3$, Fig. 11 shows closed cubic Q-Bézier curves and the global effect on the curves by altering the values of the shape parameters $\lambda_i\,(i = 1, 2, 3)$ at the same time. The closed cubic Q-Bézier curves are generated by setting $(\lambda_1, \lambda_2, \lambda_3)$ to $(1, 1, 0.8)$ (solid lines) and $(-0.1, -1, -0.1)$ (dashed lines) and $(1, -1, 1)$ (dotted lines) and $(-0.1, 1, -0.2)$ (dashdotted lines), respectively.

For $n = 4$, Figs. 12 and 13 show the quartic Q-Bézier curves and the effect on the curves by altering the values of the shape parameters $\lambda_i\,(i = 1, 2, 3, 4)$ at the same time. The quartic Q-Bézier curves are generated by setting $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

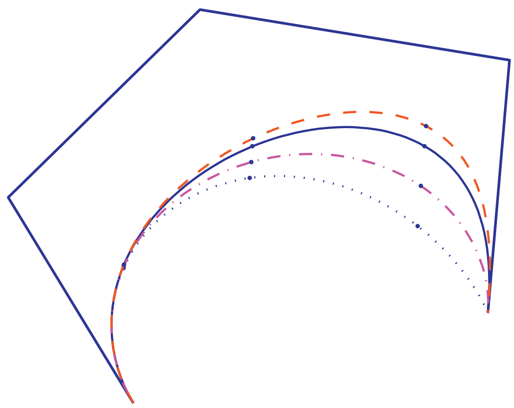

Fig. 10. The effect of the shape of quartic Q-Bézier curves by λ_4.

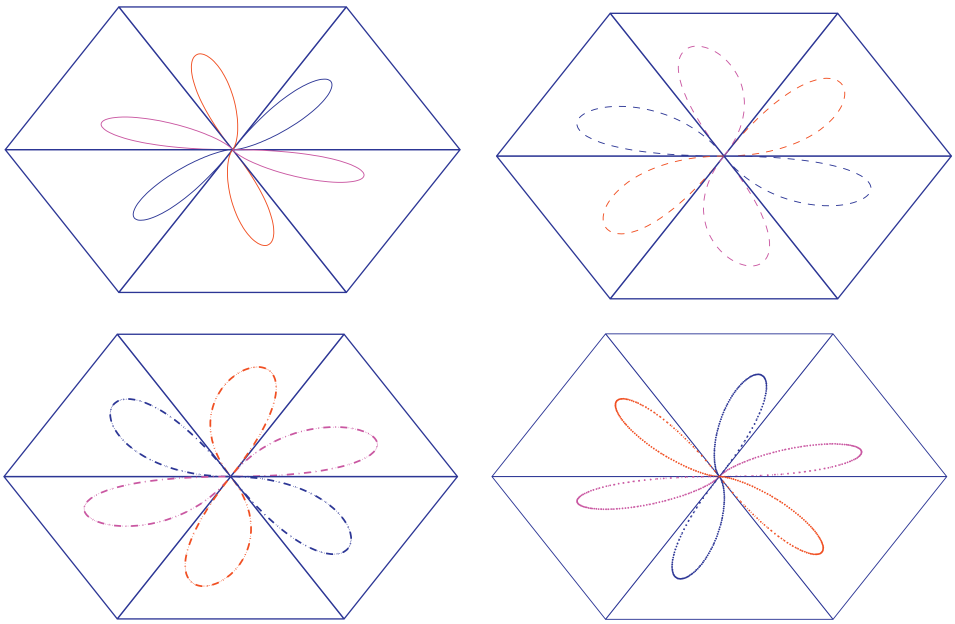

Fig. 11. The shape of closed cubic Q-Bézier curves by altering the value of $\lambda_1, \lambda_2, \lambda_3$.

to $(-4, 0, 0, 0.8)$ (solid lines) and $(-2, 1, 0, -1.8)$ (dashed lines) and $(0, -1, -2, -1)$ (dotted lines) and $(1, -2, 1, 1)$ (dashdotted lines), respectively.

5. Composite Q-Bézier curves

The condition of C^2 continuity between two Q-Bézier curves is discussed as follows.

Let a Q-Bézier curve $\boldsymbol{r}(t; \lambda_i)$ with control points $\boldsymbol{P} = (\boldsymbol{P}_0, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_n)$, $n \geqslant 2$, $\boldsymbol{P}_i \in \mathbb{R}^2$ or $\mathbb{R}^3$ be given as (3.1) and a second curve $\boldsymbol{r}^*(t; \lambda_i^*)$ with control points $\boldsymbol{P}^* = (\boldsymbol{P}_0^*, \boldsymbol{P}_1^*, \ldots, \boldsymbol{P}_m^*)$, $m \geqslant 2$, $\boldsymbol{P}_i^* \in \mathbb{R}^2$ or $\mathbb{R}^3$ by

$$\boldsymbol{r}^*(t) = \sum_{i=0}^{m} \boldsymbol{P}_i^* b_i^n(t), \quad t \in [0, 1],$$

where $\lambda_i^* \in \left[-\binom{m}{i}, \binom{m}{i-1}\right]$, $i = 1, 2, \ldots, [\frac{m}{2}]$, $\lambda_i^* \in \left[-\binom{m}{i-1}, \binom{m}{i}\right]$, $i = [\frac{m}{2}] + 1, \ldots, m$.

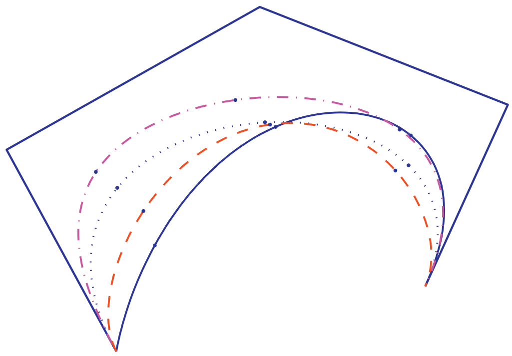

Fig. 12. The effect of the shape of C-shape quartic Q-Bézier curves by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

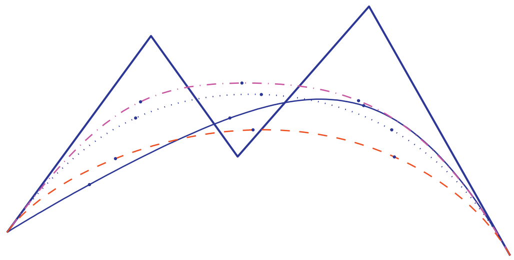

Fig. 13. The effect of the shape of M-shape quartic Q-Bézier curves by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Clearly, for the composite curve to be C^2 continuous, it is necessary and sufficient that

$$\boldsymbol{r}(1) = \boldsymbol{r}^*(0), \tag{5.1}$$

$$\boldsymbol{r}'(1) = \boldsymbol{r}^{*\prime}(0), \tag{5.2}$$

$$\boldsymbol{r}''(1) = \boldsymbol{r}^{*\prime\prime}(0), \tag{5.3}$$

For ordinary Bézier curves it is well known that the condition of continuity as follows.

Lemma 5.1. *Given two segments of Bézier curves with the control points* $\boldsymbol{P}$ *and* $\boldsymbol{P}^*$, *they are defined as*

$$\mathbb{B}(t) = \sum_{i=0}^{n} \boldsymbol{P}_i B_i^n(t), \quad t \in [0, 1],$$

and

$$\mathbb{B}^*(t) = \sum_{i=0}^{m} \boldsymbol{P}_i^* B_i^n(t), \quad t \in [0, 1],$$

then the necessary and sufficient condition of continuity is

(a) *for* C^0 *continuity*,

$$\boldsymbol{P}_n = \boldsymbol{P}_0^*; \tag{5.4}$$

(b) *for* C^1 *continuity*,

$$\boldsymbol{P}_n = \boldsymbol{P}_0^* = \frac{n}{m+n} \boldsymbol{P}_{n-1} + \frac{m}{m+n} \boldsymbol{P}_1^*; \tag{5.5}$$

(c) *for C^2 continuity*,

$$\boldsymbol{P}_2^* = \frac{n(n-1)}{m(m-1)}\boldsymbol{P}_{n-2} + \frac{1}{m(m-1)}[n(n+2m-3)(\boldsymbol{P}_n - \boldsymbol{P}_{n-1}) + m(m-1)\boldsymbol{P}_n - n(n-1)\boldsymbol{P}_{n-1}]. \tag{5.6}$$

According to the terminal properties of Q-Bézier curves, the theorem below shows the condition of continuity of the composite Q-Bézier curves.

Theorem 5.1. *Given two segments of Q-Bézier curves with the control points* $\boldsymbol{P}$ *and* $\boldsymbol{P}^*$, *then the necessary and sufficient condition of continuity is*

(a) *for C^0 continuity*,

$$\boldsymbol{P}_n = \boldsymbol{P}_0^*; \tag{5.7}$$

(b) *for C^1 continuity*,

$$\boldsymbol{P}_n = \boldsymbol{P}_0^* = \frac{(n+\lambda_n)\boldsymbol{P}_{n-1} + (m+\lambda_1^*)\boldsymbol{P}_1^*}{m+n+\lambda_n+\lambda_1^*}; \tag{5.8}$$

(c) *for C^2 continuity*,

$$\boldsymbol{P}_2^* = a\boldsymbol{P}_{n-2} + b[c(\boldsymbol{P}_n - \boldsymbol{P}_{n-1}) + 2\lambda_{n-1}\boldsymbol{P}_{n-1} + 2\lambda_2^*\boldsymbol{P}_n]; \tag{5.9}$$

where $a = \frac{n(n-1)-2\lambda_{n-1}}{m(m-1)+2\lambda_2^*}$, $b = \frac{1}{m(m-1)+2\lambda_2^*}$, $c = 4n^2 + n\lambda_n - 2n + \frac{2(n+\lambda_n)(\lambda_2^*-m)}{m+\lambda_1^*}$.

Proof. All the formulas result from straightforward computations (a) is obvious.

For (b), according to (5.2), (3.3) and (5.7),

$$(n+\lambda_n)(\boldsymbol{P}_n - \boldsymbol{P}_{n-1}) = (m+\lambda_1^*)(\boldsymbol{P}_1^* - \boldsymbol{P}_0^*), \tag{5.10}$$

(5.8) holds after simple reorganization.

For (c), from (5.8),

$$\boldsymbol{P}_1^* - \boldsymbol{P}_0^* = \frac{n+\lambda_n}{m+\lambda_1^*}(\boldsymbol{P}_n - \boldsymbol{P}_{n-1}). \tag{5.11}$$

According to (5.3) and (3.4),

$$\begin{aligned} &n(n-1)[(\boldsymbol{P}_n - \boldsymbol{P}_{n-1}) - (\boldsymbol{P}_{n-1} - \boldsymbol{P}_{n-2})] + 2n\lambda_n(\boldsymbol{P}_n - \boldsymbol{P}_{n-1}) + 2\lambda_{n-1}(\boldsymbol{P}_{n-1} - \boldsymbol{P}_{n-2}) \\ &\quad = m(m-1)[(\boldsymbol{P}_2^* - \boldsymbol{P}_1^*) - (\boldsymbol{P}_1^* - \boldsymbol{P}_0^*)] - 2m\lambda_1^*(\boldsymbol{P}_1^* - \boldsymbol{P}_0^*) + 2\lambda_2^*(\boldsymbol{P}_2^* - \boldsymbol{P}_1^*). \end{aligned} \tag{5.12}$$

Substituting (5.7), (5.8) and (5.11) into (5.12), (5.9) can be obtained. □

Remark 5.1. The condition of C^1 continuity (5.8) of Q-Bézier curves (3.1) is more flexible compared with the condition of ordinary Bézier curves (5.5). Especially, when $\lambda_n = \lambda_1^*$ the condition (5.8) is same as (5.5).

Remark 5.2. The condition of C^2 continuity (5.9) of Q-Bézier curves (3.1) is more flexible compared with the condition of ordinary Bézier curves (5.6). Especially, when $\lambda_{n-1} = \lambda_2^* = 0$ the condition (5.9) is same as (5.6).

Example 5.1. The example is shown in Fig. 14. Given a quadratic Q-Bézier curve with control points $\boldsymbol{P}_0 = \binom{0}{0}$, $\boldsymbol{P}_1 = \binom{1}{1}$, $\boldsymbol{P}_2 = \binom{2}{0}$ and $\lambda_1 = 1$, $\lambda_2 = 0.8$, and $\boldsymbol{P}_0^* = \binom{2}{0}$, $\boldsymbol{P}_3^* = \binom{1.5}{0}$, $\lambda_1^* = 0.5$, $\lambda_2^* = 1$, according to (5.8) and (5.9), a set of C^2 cubic Q-Bézier curves with different shape parameter λ_3^* is determined connected with the quadratic Q-Bézier and the other two control points $\boldsymbol{P}_1^*, \boldsymbol{P}_2^*$ can be computed as $\boldsymbol{P}_1^* = \binom{2.8}{-0.8}$, $\boldsymbol{P}_2^* = \binom{2.05}{-1.05}$.

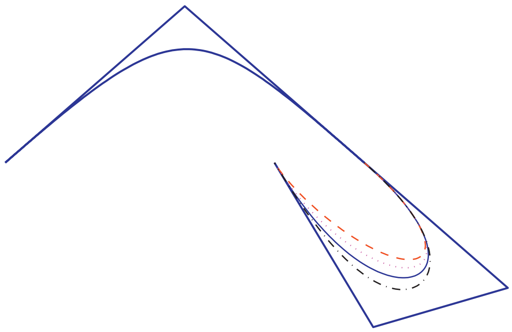

Fig. 14. The effect of the shape of resulted C^2 cubic Q-Bézier curves by λ_3^*.

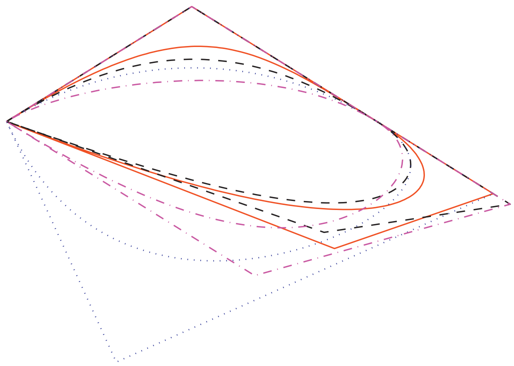

Fig. 15. The effect of the shape of resulted C^2 cubic Q-Bézier curves by shape parameters.

The resulted cubic Q-Bézier curves with control points $\boldsymbol{P}_0^*, \boldsymbol{P}_1^*, \boldsymbol{P}_2^*, \boldsymbol{P}_3^*$ and $\lambda_1^*=0.5$, $\lambda_2^*=1$ are shown in Fig. 14 by changing λ_3^* to $\lambda_3^*=0$ (solid lines) and $\lambda_3^*=-1$ (dotted lines) and $\lambda_3^*=1$ (dashdotted lines) and $\lambda_3^*=-2$ (dashed lines), respectively.

Example 5.2. The example is shown in Fig. 15. Given a quadratic Q-Bézier curve with control points $\boldsymbol{P}_0=\binom{0}{0}$, $\boldsymbol{P}_1=\binom{1}{2}$, $\boldsymbol{P}_2=\binom{2}{0}$ and $\boldsymbol{P}_0^*=\binom{2}{0}$, $\boldsymbol{P}_3^*=\binom{0}{0}$, setting the shape parameters $(\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*, \lambda_3^*)$ to $(1, 0.2, 0.5, 0, 1)$, $(0.5, -0.2, -0.5, 1, 1)$, $(-0.5, 0.2, 0.5, -1, 1)$ and $(-1, -0.2, -0.5, 1, 1)$, respectively, according to (5.8) and (5.9), a set of C^2 cubic Q-Bézier curves with different control points $\boldsymbol{P}_1^*, \boldsymbol{P}_2^*$ and shape parameters is determined connected with the quadratic Q-Bézier. The other two control points $\boldsymbol{P}_1^*, \boldsymbol{P}_2^*$ can be computed as $\binom{2.6286}{-1.2571}$, $\binom{1.7714}{-2.2095}$ and $\binom{2.72}{-1.44}$, $\binom{1.715}{-1.93}$ and $\binom{2.6286}{-1.2571}$, $\binom{0.5929}{-4.1857}$ and $\binom{2.72}{-1.44}$, $\binom{1.34}{-2.68}$ under the corresponding given shape parameters, respectively.

The resulted cubic Q-Bézier curves are shown in Fig. 15 by setting $(\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*, \lambda_3^*)$ to $(1, 0.2, 0.5, 0, 1)$(solid lines) , $(-0.5, 0.2, 0.5, -1, 1)$ (dotted lines) , $(-1, -0.2, -0.5, 1, 1)$ (dashdotted lines) and $(0.5, -0.2, -0.5, 1, 1)$ (dashed lines), respectively.

Example 5.3. The example is shown is Fig. 16. Given a cubic Q-Bézier curve with control points $\boldsymbol{P}_0=\binom{0}{0}$, $\boldsymbol{P}_1=\binom{1}{1}$, $\boldsymbol{P}_2=\binom{2}{1}$, $\boldsymbol{P}_3=\binom{3}{0.5}$ and $\lambda_1=1$, $\lambda_2=0.8$, $\lambda_3=1$, and $\boldsymbol{P}_0^*=\binom{3}{0.5}$, $\boldsymbol{P}_3^*=\binom{3}{-2}$, $\lambda_1^*=0.5$, $\lambda_2^*=1$, according to (5.8) and (5.9), a sets of C^2 cubic Q-Bézier curve with different shape parameter λ_3^* is determined connected with the cubic Q-Bézier and the other two control points $\boldsymbol{P}_1^*, \boldsymbol{P}_2^*$ can be computed as $\boldsymbol{P}_1^*=\binom{4.1429}{-0.0714}$, $\boldsymbol{P}_2^*=\binom{5.2536}{-0.9018}$.

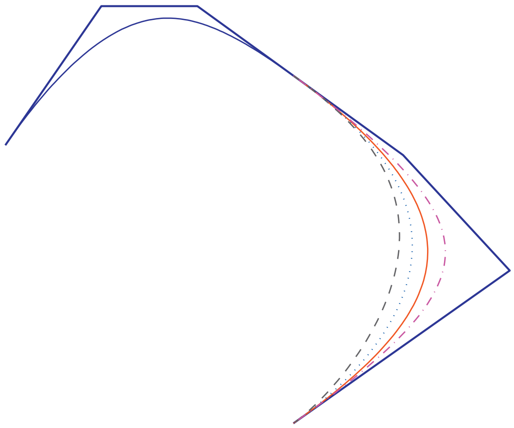

Fig. 16. The effect of the shape of resulted C^2 cubic Q-Bézier curves by λ_3^*.

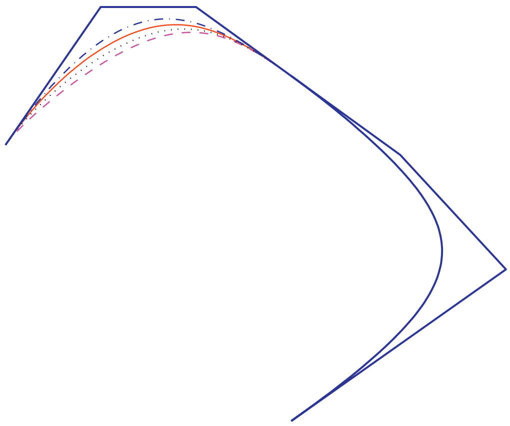

Fig. 17. The effect of the shape of C^2 composite cubic Q-Bézier curves by λ_1.

The resulted cubic Q-Bézier curves with control points $\boldsymbol{P}_0^*, \boldsymbol{P}_1^*, \boldsymbol{P}_2^*, \boldsymbol{P}_3^*$ and $\lambda_1^* = 0.5$, $\lambda_2^* = 1$ are shown in Fig. 16 by changing λ_3^* to $\lambda_3^* = 0$ (solid lines) and $\lambda_3^* = -1$ (dotted lines) and $\lambda_3^* = 1$ (dashdotted lines) and $\lambda_3^* = -2$ (dashed lines), respectively.

Example 5.4. The example is shown in Fig. 17. Given a cubic Q-Bézier curve with control points $\boldsymbol{P}_0 = \binom{0.2}{0.5}$, $\boldsymbol{P}_1 = \binom{1}{1}$, $\boldsymbol{P}_2 = \binom{2}{1}$, $\boldsymbol{P}_3 = \binom{3}{0.5}$ and $\lambda_2 = 0.8$, $\lambda_3 = 1$, and $\boldsymbol{P}_0^* = \binom{3}{0.5}$, $\boldsymbol{P}_3^* = \binom{0.2}{0.5}$, $\lambda_1^* = 0.5$, $\lambda_2^* = 1$, $\lambda_3^* = 1$, according to (5.8) and (5.9), a unique C^2 cubic Q-Bézier curve is determined connected with the cubic Q-Bézier and the other two control points $\boldsymbol{P}_1^*, \boldsymbol{P}_2^*$ can be computed as $\boldsymbol{P}_1^* = \binom{4.1429}{-0.0714}$, $\boldsymbol{P}_2^* = \binom{5.2536}{-0.9018}$.

The resulted composite cubic Q-Bézier curves with control points $\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3, \boldsymbol{P}_0^*, \boldsymbol{P}_1^*, \boldsymbol{P}_2^*, \boldsymbol{P}_3^*$ and $\lambda_2 = 0.8$, $\lambda_3 = 1$, $\lambda_1^* = 0.5$, $\lambda_2^* = 1$, $\lambda_3^* = 1$ are shown in Fig. 17 by changing λ_1 to $\lambda_1 = 0$ (solid lines) and $\lambda_1 = -1$ (dotted lines) and $\lambda_1 = 1$ (dashdotted lines) and $\lambda_1 = -2$ (dashed lines), respectively.

Example 5.5. The example is shown in Fig. 18. Given a cubic Q-Bézier curve with control points $\boldsymbol{P}_0 = \binom{0}{-0.5}$, $\boldsymbol{P}_1 = \binom{1}{1}$, $\boldsymbol{P}_2 = \binom{3}{1}$, $\boldsymbol{P}_3 = \binom{5}{1.5}$ and $\boldsymbol{P}_0^* = \binom{5}{1.5}$, $\boldsymbol{P}_3^* = \binom{3}{-1.5}$, setting the shape parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_1^*, \lambda_2^*, \lambda_3^*)$ to $(1, 0.5, 0.2, 0.5, 1, 1)$, $(1, -0.5, 0.2, 0, 0.2, 1)$, $(1, 0, -0.2, 1, -1, 1)$ and $(1, -1, -0.2, 1, -1, 1)$, respectively, according to (5.8) and (5.9), a sets of C^2 cubic Q-Bézier curve with different control points $\boldsymbol{P}_1^*, \boldsymbol{P}_2^*$ and shape

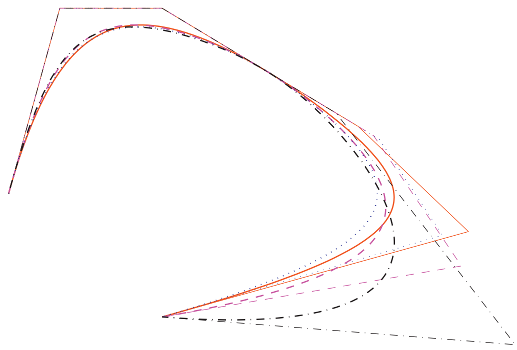

Fig. 18. Open C^2 composite cubic Q-Bézier curves with different shape parameters.

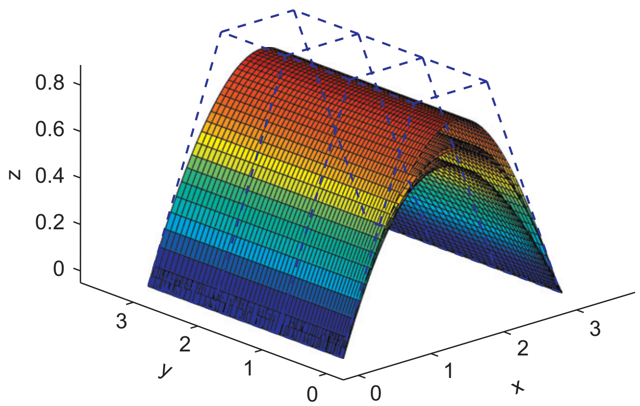

Fig. 19. The shape of bi-cubic Q-Bézier surfaces with different shape parameters.

parameters is determined connected with the given cubic Q-Bézier. The other two control points $\boldsymbol{P}_1^*, \boldsymbol{P}_2^*$ can be computed as $\begin{pmatrix} 6.8286 \\ 0.0429 \end{pmatrix}, \begin{pmatrix} 8.9857 \\ -0.8089 \end{pmatrix}$ and $\begin{pmatrix} 7.1333 \\ -0.0333 \end{pmatrix}, \begin{pmatrix} 8.8476 \\ -1.0869 \end{pmatrix}$ and $\begin{pmatrix} 7.24 \\ 8.4775 \end{pmatrix}, \begin{pmatrix} 8.4775 \\ -0.8381 \end{pmatrix}$ and $\begin{pmatrix} 6.4 \\ 0.15 \end{pmatrix}, \begin{pmatrix} 9.9 \\ -1.725 \end{pmatrix}$ under the corresponding given shape parameters, respectively.

The resulted cubic Q-Bézier curves are shown in Fig. 18 by setting $(\lambda_1, \lambda_2, \lambda_3, \lambda_1^*, \lambda_2^*, \lambda_3^*)$ to $(1, 0.5, 0.2, 0.5, 1, 1)$ (solid lines), $(1, 0, -0.2, 1, -1, 1)$ (dotted lines), $(1, -1, -0.2, 1, -1, 1)$ (dashdotted lines) and $(1, -0.5, 0.2, 0, 0.2, 1)$ (dashed lines), respectively.

Remark 5.3. From Figs. 14–17, the resulted cubic Q-Bézier curves can be locally controlled by the shape parameter λ_3^* under the control polygon keeping unchanged. A desired curve can be selected with suitable value of the shape parameter λ_3^*.

Remark 5.4. From Figs. 15 and 18, the resulted C^2 composite cubic Q-Bézier curves can be globally controlled by the shape parameters and with different control points. The shape can be adjustable. A desired curve can be selected with suitable value of the shape parameters.

6. Q-Bézier surface

Using tensor product, we can construct Q-Bézier surface

$$\boldsymbol{S}(s, t) = \sum_{i=0}^{m} \sum_{i=0}^{n} \boldsymbol{P}_{i,j} b_i^m(t) b_j^n(t), \quad 0 \leqslant s, t \leqslant 1, \tag{6.1}$$

in which $b_i^m(t)$, $b_j^n(t)$ are the generalized Bernstein basis function (2.1) and $\boldsymbol{P}_{i,j} \in \mathbb{R}^3$ is the control points. Tensor product of Q-Bézier curves has properties similar to those of tensor product of Bézier curves.

Fig. 19 shows the bi-cubic Q-Bézier surfaces and the effect on the surfaces by altering the values of the shape parameters at the same time under keeping the control net.

7. Conclusion

As mentioned above, Q-Bézier curves have all properties that ordinary Bézier curves have. However, the shape of Q-Bézier curves (3.1) can be adjusted with alerting the values of provided shape parameters, when the control polygon is maintained. And the conditions of C^2 continuity (5.8) and (5.9) are more flexible than that of ordinary Bézier curves (5.5) and (5.6). Also, because there is nearly no difference in structure between a Q-Bézier curve and a ordinary Bézier curve , it is not difficult to adopt Q-Bézier curve to a CAD/CAM system that already uses the ordinary Bézier curves.

For practical applications of Q-Bézier curves, it is clear that we need some special algorithm. Some interesting results in this area are taken into account, and they will be discussed in our future works.

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